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Numerical representation of binary relations with a multiplicative error function

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Abstract

This paper studies the case of the representation of a binary relation via a numerical function with threshold (error) depending on both compared alternatives. The error is considered to be multiplicative, its value being either directly or inversely proportional to the values of the numerical function.

For the first case, it is proved that a binary relation is a semiorder. Moreover, any semiorder can be represented in this form. In the second case, the corresponding binary relation is an interval order. © 2002 Published by Elsevier Science Inc.

Keywords: Utility function; Error function; Binary relation

1. Introduction

Rational choice paradigm lies in the basis of models of decision making, economics and psychology. The very core of this paradigm is that preferences of an individual over objects are transitive as well as her indifference relation.

In the 19th century, Fechner [12] pointed out that “the discrimination relation between stimulus is generally not transitive: this can be explained by the concept of differential threshold”. Amstrong [8–10] drew attention to the fact that an indifference relation is not transitive because the human mind is not

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necessarily capable of perfect discrimination, and introduced the notion of semiorders. A semiorder was precisely axiomatized by Luce [15] who introduced a new numerical representation which involves criterial estimates with a constant error. The limitation of these studies was that they all worked with constant errors.

Wiener [19] was the first to study theoretically the problem of evaluation of alternatives with an error that depends on one of the alternatives under pairwise comparison. His paper showed that interval order is an important concept for analysis of temporal events, each of which occurs over some time span. In this analysis, each measurement is given as an interval instead of a single point. To be more precise, event x precedes event y when x ends before y begins. Sholomov and Yudin [18] showed the relation between the problem under study and inference construction in the theory of databases. Aleskerov and Vol'skiy [7] investigated the problem of numerical representation of binary relations where the error function is dependent on both of the compared alternatives was investigated. Later in Aleskerov [5,6], Agaev and Aleskerov [4], and Aizerman and Aleskerov [3] particular cases of this problem were considered, one of them being the case where an error function is additive on its arguments. In Abbas and Vincke [1] and in Fodor and Roubens [14], the cases where the error function satisfies the triangle inequality were considered. Although these studies were significant improvements over models that used constant error, none of them tackled with multiplicative error functions without any restrictions.

This paper aims to analyze the case where the error function $\varepsilon(x,y)$ depends on both compared alternatives x and y and is multiplicative. The underlying assumption is that the error function of alternatives, that is $\varepsilon(x)$, depends on the numerical function $u(x)$. Analyzing the problem with this set of assumptions will be helpful in capturing the differences between the sensitivity levels of the preferences individuals, e.g., in various social strata. In Section 2 all preliminary notions and basic definitions are given. Section 3 contains the results on numerical representation of binary relations where the multiplicative error function depends on both of the compared alternatives x and y.

2. Preliminary notions

Consider the finite set A of alternatives. A binary relation on a set A is a set of ordered pairs (x,y) with $x \in A$ and $y \in A$. xPy and $xP^{c}y$ refer to $(x,y) \in P$ and $(x,y) \notin P$, respectively.

Definition 1. A binary relation P is said to have a numerical representation via a numerical function with error if there exist two real-valued functions $u(\cdot)$ and δ such that

$$xPy \iff u(x) - u(y) > \delta. \tag{2.1}$$

This means that there exists an insensitivity zone (or measurement error) δ in which these alternatives can be considered as indifferent in terms of choice even if their utilities are different. For example, although the distinction between one and three cubes of sugar in a coffee makes a difference in taste, we would not be able to differentiate between the tastes of one and two cubes or two and three cubes. In other words, we are indifferent between n cubes and $n+1$ cubes, yet have a definite preference between one cube and 10 cubes.

Definitions of special types of binary relations – those of interval order and semiorder (for detailed studies see, e.g., [11,13,15,17]) are provided below.

Definition 2. An irreflexive binary relation P is called an interval order if

$$\forall x, y, z, w \in A \quad xPy \text{ and } zPw \;\Rightarrow\; xPw \text{ or } zPy$$

an interval order which satisfies the condition

$$\forall x, y, z, w \in A \quad xPy \text{ and } yPz \;\Rightarrow\; xPw \text{ or } wPz$$

is called a semiorder.

It can be easily seen that the class of semiorders is a proper subset of the class of interval orders.

Depending on the form of the function δ one can obtain different types of binary relations P in (2.1): P is a semiorder if and only if the constant error function δ is non-negative, $\delta = \text{constant} \geqslant 0$ [15]; P is an interval order if and only if the error function $\delta(\cdot)$ is a non-negative real-valued function defined on the set A, $\delta = \varepsilon(x) \geqslant 0$ [13]. If the assumption that δ is non-negative is relaxed, then one can cover a more general class of relations – coherent bi-orders and bi-orders, respectively (see [11]).

In [1–5,7], the model is studied in which ε in (2.1) depends on both of the compared alternatives x and y. This means that the error function $\delta(\cdot,\cdot)$ is a real-valued function defined on the set $A \times A$. For this case (2.1) can be expressed as

$$xPy \iff u(x) - u(y) > \delta(x, y). \tag{2.2}$$

It has been shown that any acyclic binary relation can be represented in this way if and only if δ is non-negative and any binary relation has such a numerical representation if and only if δ is not restricted. For the case when $\delta(\cdot,\cdot)$ additively depends on $\varepsilon(x)$ and $\varepsilon(y)$, i.e.,

$$\delta(x, y) = \varepsilon(x) + \varepsilon(y)$$

the corresponding P is an interval order (see [3,5]).

3. The case of a multiplicative error function

Now consider the case where the error function $\delta(\cdot,\cdot)$ is multiplicative, i.e.,

$$\delta(x,y) = \varepsilon(x)\cdot\varepsilon(y)$$

and, moreover, the function ε is dependent in a different way on the value of the numerical function $u(\cdot)$.

Two such cases are studied. In one of them, the error value decreases when its numerical value increases. This corresponds to the case where alternatives with small numerical values are considered to be similar. The second case is opposite-error value of an alternative increases along with its numerical values, which corresponds to the case where alternatives with high utilities are considered to be similar. We can exemplify these two cases in the following way: In the first case, an affluent man does not feel the need to differentiate between the prices in the supermarket and the local bazaar, being both comparatively cheap. On the contrary, in the second case, a poor man would not to be able to distinguish between two luxury cars because of their inaccessibility.

Theorem 3.1. *Let P have a numerical representation with an error of type* (2.2). *Here the function $u(\cdot)$ is positive, $\delta(\cdot,\cdot)$ is multiplicative, i.e., $\forall x,y,\ \delta(x,y) = \varepsilon(x)\cdot\varepsilon(y)$, and the function $\varepsilon(x)$ depends on $u(x)$ in such a way that $\varepsilon(x) = \alpha/u(x)$ with $\alpha > 0$. Then P is an interval order.*

Proof.

(i) P is irreflexive. Since $\delta(x,x) = \varepsilon(x)\cdot\varepsilon(x) = (\alpha/u(x))(\alpha/u(x)) = \alpha^2/(u^2(x)) > 0$, then $\delta(x,x) > 0 = u(x) - u(x)$. Thus xP^cx.

(ii) P satisfies strong intervality. Assume the contrary $xPy \wedge zPw \wedge xP^cw \wedge zP^cy$. Then

$$u(x) - u(y) > \frac{\alpha}{u(x)}\frac{\alpha}{u(y)}, \tag{3.1}$$

$$u(z) - u(w) > \frac{\alpha}{u(z)}\frac{\alpha}{u(w)}, \tag{3.2}$$

$$u(x) - u(w) \leqslant \frac{\alpha}{u(x)}\frac{\alpha}{u(w)}, \tag{3.3}$$

$$u(z) - u(y) \leqslant \frac{\alpha}{u(z)}\frac{\alpha}{u(y)}. \tag{3.4}$$

Adding (3.1) and (3.2), and (3.3) and (3.4), we obtain

$$\frac{\alpha}{u(x)}\frac{\alpha}{u(y)} + \frac{\alpha}{u(z)}\frac{\alpha}{u(w)} < \frac{\alpha}{u(z)}\frac{\alpha}{u(y)} + \frac{\alpha}{u(x)}\frac{\alpha}{u(w)}.$$

Multiply both sides by $u(x)u(y)u(z)u(w)/\alpha^2$, we obtain

$$u(x)u(y) + u(z)u(w) < u(x)u(w) + u(y)u(z).$$

Then $u(z)u(w) - u(y)u(z) < u(x)u(w) - u(x)u(y)$, so

$$u(z)(u(w) - u(y)) < u(x)(u(w) - u(y)). \tag{3.5}$$

Moreover, (3.1) and (3.3) imply

$$u^2(x)u(y) - u(x)u^2(y) > \alpha^2 \geqslant u^2(x)u(w) - u(x)u^2(w). \tag{3.6}$$

Inequalities (3.2) and (3.4) imply

$$u^2(z)u(w) - u(z)u^2(w) > \alpha^2 \geqslant u^2(z)u(y) - u(z)u^2(y). \tag{3.7}$$

Consider three possible cases in (3.5):

1. $u(w) > u(y)$,
2. $u(w) < u(y)$,
3. $u(w) = u(y)$.

Case 1: $u(w) > u(y) \Rightarrow u(x) > u(z)$,

From (3.6) $u^2(x)(u(w) - u(y)) < u(x)(u^2(w) - u^2(y))$, which implies

$$u^2(x)(u(w) - u(y)) < u(x)(u(w) - u(y))(u(w) + u(y)),$$

then

$$u(x) < u(w) + u(y). \tag{3.8}$$

From (3.7) $u^2(z)(u(w) - u(y)) > u(z)(u^2(w) - u^2(y))$, which implies $u^2(z)(u(w) - u(y)) > u(z)(u(w) - u(y))(u(w) + u(y))$, then

$$u(z) > u(w) + u(y). \tag{3.9}$$

From (3.8) and (3.9) we get $u(z) > u(x)$ which contradicts $u(z) < u(x)$ obtained in (3.5).

Case 2: $u(w) < u(y) \Rightarrow u(x) < u(z)$.

From (3.6), we obtain $u^2(x)(u(w) - u(y)) < u(x)(u^2(w) - u^2(y))$, which implies

$$u(x) > u(w) + u(y). \tag{3.10}$$

From (3.7), we obtain $u^2(z)(u(w) - u(y)) > u(z)(u^2(w) - u^2(y))$, which implies

$$u(z) < u(w) + u(y). \tag{3.11}$$

From (3.10) and (3.11) we get $u(z) < u(x)$ which contradicts $u(z) > u(x)$.

Case 3: $u(w) = u(y)$.

From (3.6), we get $u^2(x)u(y) - u(x)u^2(y) > \alpha^2 \geqslant u^2(x)u(y) - u(x)u^2(y)$ which contradicts $\alpha > 0$. $\square$

Theorem 3.1 shows the conditions for P to be an interval order. However, we could not prove the inverse statement which remains an open problem.

Example 1. An interval order P having a numerical representation as stated in Theorem 3.1 is not in general a semiorder. This can be shown by the following example: Let $u(x) = 1, u(y) = 0.8, u(z) = 0.5, u(w) = 0.01$ and $\alpha = 0.1$. Then it is easily seen that $xPyPz$ but $xP^c w$ and $wP^c z$.

Theorem 3.2. *Let P have a numerical representation with error of type* (2.2). *Here the function $u(\cdot)$ is positive, $\delta(\cdot,\cdot)$ is multiplicative, i.e., $\forall x, y,\ \delta(x,y) = \varepsilon(x)\cdot\varepsilon(y)$, and the function $\varepsilon(x)$ depends on $u(x)$ such that $\varepsilon(x) = \alpha u(x)$ with $\alpha > 0$. Then P is a semiorder.*

Proof.

(i) P is irreflexive. Since $\varepsilon(x,x) = \varepsilon(x)\cdot\varepsilon(x) = \alpha^2 u^2(x) > 0$, then $\delta(x,x) > 0 = u(x) - u(x)$. Thus $xP^c x$.

(ii) P satisfies strong intervality. Assume on the contrary $xPy \wedge zPw \wedge xP^c w \wedge zP^c y$. Then

$$u(x) - u(y) > \alpha^2 u(x)u(y), \tag{3.12}$$

$$u(z) - u(w) > \alpha^2 u(z)u(w), \tag{3.13}$$

$$u(x) - u(w) \leqslant \alpha^2 u(x)u(w), \tag{3.14}$$

$$u(z) - u(y) \leqslant \alpha^2 u(z)u(y). \tag{3.15}$$

Inequalities (3.12) and (3.14) imply that

$$u(y) < \frac{u(x)}{1+\alpha^2 u(x)} \leqslant u(w) \quad\Rightarrow\quad u(y) < u(w).$$

Inequalities (3.13) and (3.15) imply that

$$u(w) < \frac{u(z)}{1+\alpha^2 u(z)} \leqslant u(y) \quad\Rightarrow\quad u(w) < u(y),$$

a contradiction.

(iii) P is semitransitive. Assume on the contrary $xPy \wedge yPz \wedge xP^c w \wedge wP^c z$. Then

$$u(x) - u(y) > \alpha^2 u(x)u(y), \tag{3.16}$$

$$u(y) - u(z) > \alpha^2 u(y)u(z), \tag{3.17}$$

$$u(x) - u(w) \leqslant \alpha^2 u(x)u(w), \tag{3.18}$$

$$u(w) - u(z) \leqslant \alpha^2 u(w)u(z). \tag{3.19}$$

Inequalities (3.16) and (3.18) imply that

$$u(y) < \frac{u(x)}{1+\alpha^2 u(x)} \leqslant u(w).$$

Then $u(y) < u(w)$, add $\alpha^2 u(w)u(y)$ to both sides,

$$\alpha^2 u(w)u(y) + u(y) < u(w) + \alpha^2 u(w)u(y),$$

from this, we obtain

$$\frac{u(y)}{1+\alpha^2 u(y)} \leqslant u(z) \leqslant \frac{u(w)}{1+\alpha^2 u(w)}.$$

At the same time, (3.17) and (3.19) imply that

$$\frac{u(y)}{1+\alpha^2 u(y)} \geqslant u(z) \geqslant \frac{u(w)}{1+\alpha^2 u(w)},$$

i.e., we obtain a contradiction. □

Theorem 3.3. *Any semiorder P has a numerical representation with error of type* (2.2). *Here the function* $u(\cdot)$ *is positive,* $\delta(\cdot,\cdot)$ *is multiplicative, i.e.,* $\forall x, y$, $\delta(x,y) = \varepsilon(x)\cdot\varepsilon(y)$, *and the function* $\varepsilon(x)$ *depends on* $u(x)$ *such that* $\varepsilon(x) = \alpha u(x)$ *with* $\alpha > 0$.

Proof. Before we start to prove this, let us construct the partitions that define the structure of an interval order (see, e.g. [16]).

Strong intervality condition ($\forall x, y, z, w \in A$, xPy and $zPw \Rightarrow xPw$ or zPy) implies that $\forall x, y \in A$, $L(x) \subseteq L(y)$ or $L(y) \subseteq L(x)$, where $L(x)$ is the lower contour set of x with respect to P, i.e., $L(x) = \{y \in A \mid xPy\}$. Irreflexivity indicates that there is a chain with respect to the lower contour sets, i.e., relabel elements of A, $|A| = n$ such that $L(x_i) \subseteq L(x_j)$ for all $1 \leqslant i \leqslant j \leqslant n$. Moreover, we can have strict inclusions such that there exists $s \leqslant n$ such that $\emptyset = L(x_1) \subset L(x_2) \cdots L(x_{s-1}) \subset L(x_s)$, where $\{x_1, x_2, \ldots, x_s\} \subseteq A$.

Define

$$I_k = \{x \in A \mid L(x_k) = L(x)\} \quad k = 1, \ldots, s.$$

I_k is not empty for any k since $x_k \in I_k$ by construction. Clearly, the system $\{I_k\}_1^s$ is a partition of the set A, i.e., $\bigcup_{k=1}^s I_k = A, I_k \cap I_k = \emptyset$ when $k \neq l$. Now construct another family of non-empty sets $\{J_m\}_1^s$, as follows:

$$\begin{aligned}
J_1 &= L(x_2) \setminus L(x_1),\\
J_2 &= L(x_3) \setminus L(x_2),\\
&\vdots\\
J_{s-1} &= L(x_s) \setminus L(x_{s-1}),
\end{aligned}$$

$$J_s = A \setminus \bigcup_{m=1}^{s-1} J_m.$$

Clearly, the system $\{J_m\}_1^s$ is a partition of the set A, i.e., $\bigcup_{m=1}^{s} J_m = A$, $J_k \cap J_m = \emptyset$ when $k \neq m$. Then any interval order P can be represented as

$$P = \bigcup_{k=2}^{s} \left[I_k \times \bigcup_{m=1}^{k-1} J_m \right]$$

with the restriction

$$I_k \subseteq \bigcup_{m=k}^{s} J_m.$$

Now define $\{Z_{k,m}\}_{k=1,\dots,s,\, m=1,\dots,s}$ s.t. $Z_{k,m} = I_k \cap J_m$. Let P_B be a binary relation on $B \subset A$ such that $P_B = P \cap (B \times B)$.

Claim 1. *If P is a semiorder on A, then P_B is also a semiorder on $B \subset A$.*

Proof of Claim. Clearly $P_B \subset P$.

$$\begin{aligned} \forall x, y, z, w \in B \quad xP_B y \wedge zP_B w \quad &\Rightarrow \quad xPy \wedge zPw \quad \text{since } P_B \subset P \\ &\Rightarrow \quad xPw \vee zPy \quad \text{since } P \text{ is a semiorder} \\ &\Rightarrow \quad xP_B w \vee zP_B y \quad \text{since } x, y, z, w \in B \end{aligned}$$

and

$$\begin{aligned} \forall x, y, z, w \in B \quad xP_B yP_B z \quad &\Rightarrow \quad xPyPz \quad \text{since } P_B \subset P \\ &\Rightarrow \quad xPw \vee wPz \quad \text{since } P \text{ is a semiorder} \\ &\Rightarrow \quad xP_B w \vee wP_B z \quad \text{since} x, y, z, w \in B. \end{aligned}$$

Therefore P_B is a semiorder. □

We will prove the following lemma that will be used in the proof of the theorem.

Lemma 1. *Let P be a semiorder which has a numerical representation as stated in the theorem. Then*
(i) $x \in Z_{k,m} \wedge y \in Z_{k,m-1} \Rightarrow u(x) > u(y)$ *and*
(ii) $x \in Z_{k,m} \wedge y \in Z_{k-1,m} \Rightarrow u(x) > u(y)$.

Proof of Lemma.
(i) If $y \notin Z_{k,m}$ then $\exists z$

$$zPy \wedge zP^c x. \tag{3.20}$$

By (3.20),

$$\begin{aligned} & u(z) - u(y) > \alpha^2 u(z)u(y) \wedge u(z) - u(x) \leqslant \alpha^2 u(x)u(y) \\ & \Rightarrow \quad u(z) > u(y)(1 + \alpha^2 u(z)) \wedge u(z) \leqslant u(x)(1 + \alpha^2 u(z)) \\ & \Rightarrow \quad u(x) - u(y) > 0. \end{aligned}$$

(ii) If $y \notin Z_{k,m}$ then $\exists z$

$$xPz \wedge yP^c z. \tag{3.21}$$

By (3.21),

$$\begin{aligned} & u(x) - u(z) > \alpha^2 u(z)u(x) \wedge u(y) - u(z) \leqslant \alpha^2 u(z)u(y) \\ & \Rightarrow \quad u(z) < u(x)(1 - \alpha^2 u(z)) \wedge u(z) \geqslant u(y)(1 - \alpha^2 u(z)) \end{aligned}$$

we know that xPz. It means that

$$\begin{aligned} u(x) - u(z) > \alpha^2 u(z)u(x) \quad & \Rightarrow \quad 0 < u(z) < u(x)(1 - \alpha^2 u(z)) \\ & \Rightarrow \quad (1 - \alpha^2 u(z)) > 0 \quad \text{since } 0 < u(x) \\ & \Rightarrow \quad u(y) \leqslant u(z)/(1 - \alpha^2 u(z)) < u(x) \\ & \Rightarrow \quad u(x) - u(y) > 0. \qquad \square \end{aligned}$$

By induction,

$|A| = 1, A = \{x\}$. Define $u(x)$ is equal to 1 and $\alpha = 1$. Hence any semiorder P on A has a numerical representation with error of type (2.2). Here the function $u(\cdot)$ is positive, $\delta(\cdot,\cdot)$ is multiplicative, i.e., $\forall x, y,\ \delta(x,y) = \varepsilon(x) \cdot \varepsilon(y)$, and the function $\varepsilon(x)$ depends on $u(x)$ such that $\varepsilon(x) = \alpha u(x)$ with $\alpha > 0$.

$|A| = n - 1$. Assume that any semiorder P on A has a numerical representation which satisfies all conditions which are given above and if x cannot beat y then $u(x) - u(y) < \alpha^2 u(x)u(y)$. We called the semiorder P has a strict numerical representation with error of type (2.2).

Show that any non-trivial semiorder P on A which has n elements has a strict numerical representation which satisfies all the conditions given above.

Take a semiorder P on A such that $|A| = n$.

If there is a $Z_{\overline{k},\overline{m}}$ which has two elements, we can erase one of these element, called it x, from P to get P_B. In this case, $I_k^B = I_k$ and $J_m^B = J_m$ for all $m, k \leqslant s$ where I_k^B and J_m^B are the partitions with respect to P_B. We know that P_B has a strict representation by the induction step. Then set the utility value of element x which is erased equal to the other element which belongs to the same partition with x. To be more precise, If $\exists \overline{k}, \overline{m} \in \{1, 2, \ldots, s\}$ such that $|Z_{\overline{k},\overline{m}}| > 1$ then consider a binary relation P_B on $B = A \setminus \{\overline{x}\}$ where $\overline{x} \in Z_{\overline{k},\overline{m}}$. We know that

P_B is a semiorder by the claim. By induction step, P_B on B has a strict numerical representation with error of type (2.2). There exists a positive function $u^B(\cdot)$, a $\delta(\cdot,\cdot)$ multiplicative, i.e., $\forall x,y,\ \delta^B(x,y) = \varepsilon^B(x)\cdot\varepsilon^B(y)$, and a function $\varepsilon^B(x)$ which depends on $u^B(x)$ such that $\varepsilon^B(x) = \alpha u^B(x)$ with $\alpha^B > 0$. We need to find a function $u(\cdot)$ and $\alpha > 0$ to represent P. Define $u(\cdot); \forall x \in B,\ u(x) = u^B(x)$ and $u(\bar{x}) = u^B(y)$ where $\bar{x} \neq y \in Z_{\bar{k},\bar{m}}$. By assuming $\alpha = \alpha^B$, we can easily show that P on A has a strict numerical representation that satisfies all the conditions given above.

So we will assume $\forall k,m \in \{1,2,\ldots,s\}$ such that $|Z_{k,m}| \leqslant 1$. In other words, there are no two elements which have the same upper and lower contour sets. When we erase the element of $Z_{1,1}$, denote it by $x_1 \in Z_{1,1}$, from P to get P_B where $B = A \setminus \{x_1\}$. Thus the number of I_k^B and J_m^B decrease by 1. Next step shows what the relationships among I_k^B, I_k, J_m^B and J_m are.

Let us derive some equalities about $Z_{k,m}^B$ for all $k,m \in \{1,2,\ldots,s-1\}$ which will be useful later:

$$I_k^B = I_{k+1} \quad \text{for all } 1 < k \leqslant s-1, \tag{3.22}$$

$$I_1^B = (I_1 \cup I_2) \setminus \{x_1\} \quad \text{for all } k = 1 \text{ where } x_1 \in Z_{1,1}, \tag{3.23}$$

$$J_m^B = J_{m+1} \quad \text{for all } m \leqslant s-1. \tag{3.24}$$

Proof. We have $\emptyset = L(x_1) \subset \{x_1\} = L(x_2)\cdots L(x_{s-1}) \subset L(x_s)$ since we assumed $\forall k,m \in \{1,2,\ldots,s\}$ such that $|Z_{k,m}^P| \leqslant 1$. After deleting $x_1 \in Z_{1,1}$, it becomes $\emptyset = L(x_1)\setminus\{x_1\} = L(x_2)\setminus\{x_1\} \subset L(x_3)\setminus\{x_1\}\cdots L(x_{s-1})\setminus\{x_1\} \subset L(x_s)\setminus\{x_1\}$. Rename elements of B such that $L^B(y_1) \subset L^B(y_2)\cdots L^B(y_{s-2}) \subset L^B(y_{s-1})$ since $L(x_1)\setminus\{x_1\} = L(x_2)\setminus\{x_1\} = \emptyset$. It means that the number of I_k^B's decreases by 1.

$$\begin{aligned} \forall x \in I_k^B \quad &\Longleftrightarrow \quad L^B(x) = L^B(y_k) = L(x_{k+1}) \\ &\Longleftrightarrow \quad x \in I_{k+1} \quad \text{for all } 1 < k \leqslant s-1. \end{aligned}$$

When

$$\begin{aligned} k = 1,\ x \in I_1^B \quad &\Longleftrightarrow \quad \emptyset = L^B(x) = L^B(y_1) = L(x_1) = L(x_2)\setminus\{x_1\} \\ &\Longleftrightarrow \quad x \in (I_1 \cup I_2)\setminus\{x_1\}. \end{aligned}$$

To prove the last statement,

$$\begin{aligned} \forall m \leqslant s-2 \quad & J_m^B = L^B(y_{m+1}) \setminus L^B(y_m) \\ \Rightarrow\ & J_m^B = (L(x_{m+2})\setminus\{x_1\}) \setminus (L(x_{m+1})\setminus\{x_1\}) \\ & = L(x_{m+2}) \setminus L(x_{m+1}) = J_{m+1}. \end{aligned}$$

We have $J^B_m = J_{m+1}$

$$\forall m < s-1 \;\Rightarrow\; \bigcup_{m=1}^{s-2} J^B_m = \bigcup_{m=2}^{s-1} J_m$$
$$\Rightarrow\; J^B_{s-1} = (A \setminus \{x_1\}) \setminus \bigcup_{m=1}^{s-2} J^B_m$$
$$= (A \setminus \{x_1\}) \setminus \bigcup_{m=2}^{s-1} J_m = A \setminus \bigcup_{m=1}^{s-1} J_m = J_s.$$

Eqs. (3.22)–(3.24) imply that $Z^B_{k,m} = Z_{k+1,m+1}$ for all $k,m \in \{2,\dots,s-1\}$ and $\max_{k,m\in\{1,2,\dots,s-1\}} |Z^B_{k,m}| \leqslant 2$ since $|Z_{k,m}| \leqslant 1$ for all $k,m \in \{1,2,\dots,s\}$. Moreover, $Z^B_{k,m}$ may differ from $Z_{k+1,m+1}$ only if k is equal to 1.

By there are two possible cases:

1. $\exists \overline{m} \in \{1,2,\dots,s-1\}$ such that $|Z^B_{1,\overline{m}}| = 2$,
2. $\forall \overline{m} \in \{1,2,\dots,s-1\}$ such that $|Z^B_{1,\overline{m}}| \leqslant 1$.

Case 1: Assume $\exists \overline{m} \in \{1,2,\dots,s-1\}$ such that $|Z^B_{1,\overline{m}}| = 2$. It implies that $|Z_{1,\overline{m}+1}| = |Z_{2,\overline{m}+1}| = 1,\ t \in Z_{1,\overline{m}+1}$ and $y \in Z_{2,\overline{m}+1}$. We want to show that $\forall m > \overline{m}+1,\ Z_{1,m} = \emptyset$. Assume there exists an $m > \overline{m}+1$ such that $|Z_{1,m}| = 1$, $z \in Z_{1,m}$. Take $w \in I_m$, we know that $wPyPx_1$ (since x_1 is in J_1 and y is in $J_{\overline{m}+1}$), $wP^c z$ (since $w \in I_m$ and z is in J_m) and $zP^c x_1$ (since z is in I_1). It contradicts that P is a semiorder. Therefore, $|Z^B_{1,\overline{m}}| = 2$ implies $|Z_{1,m}| = 0$ for all $m > \overline{m}+1$. This fact shows that if there exist $\overline{m}$ such that $|Z^B_{1,\overline{m}}| = 2$ then it must be unique. It means that $\forall m > \overline{m}+1,\ Z_{1,m} = \emptyset$ and $Z_{2,m} = \emptyset$ for all $m < \overline{m}+1$. To prove the latter statement, assume there exists an $m < \overline{m}+1$ such that $z \in Z_{2,m}$. Take $w \in J_{\overline{m}+1}$, we know that $wPzPx_1$ (since x_1 is in J_1 and z is in I_2), $wP^c t$ (since $w \in I_{\overline{m}+1}$ and t is in $J_{\overline{m}+1}$) and $tP^c x_1$ (since t is in I_1).

We know that P_B is a semiorder by Claim 1. So there exists a positive function $u^B(\cdot)$, a $\delta(\cdot,\cdot)$ multiplicative, i.e., $\forall x,y,\ \delta^{\mathrm{B}}(x,y) = \varepsilon^B(x)\cdot\varepsilon^B(y)$, and a function $\varepsilon^B(x)$ which depends on $u^B(x)$ such that $\varepsilon^B(x) = \alpha^B u^B(x)$ with $\alpha^B > 0$. They represent P_B on B.

So $\exists! \overline{m} \in \{1,2,\dots,s-1\}$ such that $|Z^B_{1,\overline{m}}| = 2$.

Let t,y be the elements of $Z^B_{1,\overline{m}}$, i.e., $t \in Z_{1,\overline{m}+1}$ and $y \in Z_{2,\overline{m}+1}$. Assume $u^B(t) = u^B(y)$ since $t,y \in Z^B_{1,\overline{m}}$, otherwise set the utility value of y is equal to the maximum of $u^B(t)$ and $u^B(y)$.

We want to define $\psi(\cdot)$ on A such that $\psi(x) = u(x)/(1+\alpha^2 u(x))$. It can be easily seen that if $\psi(x)$ is less (greater) than $u(y)$ then $xP^c y$ (xPy). Moreover $\psi(x) < u(x)$ for all $x \in A$.

Let z be the element of $Z^B_{\overline{m},m}$ where m is the highest integer such that the intersection of $I^B_{\overline{m}}$ and J^B_m is not empty. We know that z is different from y since $n > 2$. Since z cannot beat y and P_B on B has a strict numerical representation, $u^B(z) - u^B(y) < (\alpha^B)^2 u^B(z)u^B(y)$. This implies $u^B(z)/(1+(\alpha^B)^2 u^B(z))$ is strictly less than $u^B(y)$, in other words $\psi^B(z) < u^B(y)$. Define

$$u(x) = \begin{cases} u^B(x) & \text{if } x \in B \setminus \{t\}, \\ \delta & \text{if } x = t, \\ \gamma & \text{if } x = x_1, \end{cases}$$

where $\psi^B(z) < \delta < u^B(y)$ and $\psi^B(y) > \gamma < \delta/(1 + (\alpha^B)^2\delta)$. To be more precise,

$$\psi(t) < u(x_1) < \psi(y) \quad \text{and} \quad \psi(z) < u(t) < u(y). \tag{3.25}$$

We want to show that $u(\cdot)$ strictly represents P on A when $\alpha = \alpha^B$. We do not need to check elements of $B \setminus \{t\}$ since nothing has changed. Let us start with x_1.

If $w \in U(x_1), u^B(y) < u^B(w)$ since $y = \arg\min_{w\in U(x_1)} u^B(w)$

$$\Rightarrow u(x_1) < \psi(y) \leqslant \psi(w) \quad \text{since (3.25)}$$

$$\Rightarrow u(w) - u(x_1) > \alpha^2 u(x_1)u(w)$$

$$\Rightarrow wPx_1.$$

If $w \in A \setminus U(x_1), u^B(y) > u^B(w)$ by lemma. Since yP^cw, we have $u(w) > \psi(y)$. By (3.25) $u(x_1) < \psi(y) < u(w)$. Hence x_1P^cw.

If $w \in U(t)$, we know that $U(t) = U(y)$ since $t \in Z_{1,\overline{m}+1}$ and $y \in Z_{2,\overline{m}+1}$. We have $u(y) < \psi(w)$ since $wPy \Rightarrow u(t) < u(y) < \psi(w)$. Therefore wPt.

If $w \in A \setminus U(t)$, then $\psi(w) < u(t)$ and $\psi(x_1) < u(x_1) < \psi(y) < u(t)$ by definition of $u(t)$ and (3.25). Hence wP^ct. And we have $\psi(y) < u(w)$ since yP^cw. By (3.25) $\psi(t) < u(x_1) < \psi(y) < u(w)$. Hence tP^cw.

Case 2: Assume $\forall m \in \{1, 2, \ldots, s-1\}$ such that $|Z^B_{1,m}| \leqslant 1$. Find a real number γ such that $\psi^B(\underline{z}) < \gamma < \psi^B(\overline{z})$ where $\overline{z} = \arg\min_{w\in U(x_1)} u^B(w)$ and $\underline{z} = \max_{w\in A\setminus U(x_1)} u^B(w)$. Define

$$u(x) = \begin{cases} u^B(x) & \text{if } x \in B, \\ \gamma & \text{if } x = x_1 \end{cases}$$

and $\alpha = \alpha^B$. Similar to Case 1, P on A has a strict numerical representation that satisfies all the conditions given above. □

Example 2. Let $A = \{a, b, c, d\}$ and consider the interval order $P = \{(a, b), (b, d), (a, d)\}$. It cannot have a numerical representation with error of type (2.2) where the function $u(\cdot)$ is positive, $\delta(\cdot,\cdot)$ is multiplicative, i.e., $\forall x, y$, $\delta(x, y) = \varepsilon(x)\cdot\varepsilon(y)$, and the function $\varepsilon(x)$ which depends on $u(x)$ such that $\varepsilon(x) = \alpha u(x)$ with $\alpha > 0$. Since P is an irreflexive binary relation, without loss of generality we can assume that the utility value of each alternative is non-negative:

$$bPd \wedge c \text{ not } Pd \iff u(b) - u(d) > \alpha^2 u(b)u(d)$$

and

$$u(c) - u(d) \leqslant \alpha^2 u(c)u(d) \quad \leftrightarrow \quad \frac{u(b) - u(d)}{\alpha^2 u(b)} > u(d) \geqslant \frac{u(c) - u(d)}{\alpha^2 u(c)}$$
$$\iff \quad u(b) > u(c).$$

This implies that $u(a) - u(b) < u(a) - u(c)$. We have $\alpha^2 u(a)u(b) < \alpha^2 u(a)u(c)$ since $aPb \wedge aP^c c$. So we find $u(b) < u(c)$ which contradicts with the previous result.

Remark 1. It can be easily shown that any weak order (semiorder which satisfies negative transitivity condition: i.e., $\forall x, y, z \in A,\ xP^c yP^c z \Rightarrow xP^c z$) has the representation given by (2.2) with a multiplicative error function and $\varepsilon(x) = \alpha u(x)$ or $\varepsilon(x) = \alpha / u(x), \alpha > 0$. Indeed, let us consider the error of the form $\varepsilon(x) = \alpha u(x)$. Weak order P is defined by the partition $\{Z_m\}_1^n$ such that

$$xPy \quad \text{iff } x \in Z_i, y \in Z_j \text{ and } i > j.$$

If we choose α to be $\alpha = 1/n$, and $u(x)$ to be equal to i if $x \in Z_i$, then for two elements in different classes Z_i and Z_j, ε will be less than 1, and for two elements from the same class ε will be equal to 1.

For the second type of error function one can choose α to be equal to 1. It can be shown that this function ε satisfies the necessary requirement.

4. Conclusions

It has been proven that a binary relation has a numerical representation with a multiplicative error function, its value being inversely proportional to the value of the numerical function when it is an interval order. The form of the error function here corresponds to the case where we cannot differentiate between the alternatives with small numerical values.

Another form the error function can take is that the value of the error varies proportionally with the numerical values. In this case stronger results can be stated. The binary relation under question becomes a semiorder, and any semiorder can be represented in this form. It is worth noting that in the case of an additive error function the following result is obtained [7]: the binary relation is represented via an additive error function if and only if it is an interval order.

Using the results above, it is now possible to represent numerically situations in which the preferences of the actors can be indifferent because of their specific circumstances that result in varying forms of rationality.

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References

[1] M. Abbas, Ph. Vincke, Preference structures and threshold models, J. Multi-Crit. Decision Anal. 2 (1993) 171–178.

[2] M. Abbas, Contribution au rapprochment de la théorie des graphes et de l'aide à la décision: graphes parfaits et modéles de préférence, Ph.D. Thesis, Université Libre de Bruxelles, Belgium, 1994.

[3] M. Aizerman, F. Aleskerov, Theory of Choice, Elsevier, North-Holland, Amsterdam, 1995.

[4] R. Agaev, F. Aleskerov, Interval choice: classic and general cases, Math. Soc. Sci. 26 (1993) 249–272.

[5] F. Aleskerov, Binary relations, numerical comparisons with errors and rationality conditions for choice, Social Science Working Paper no. 812, California Institute of Technology, 1992.

[6] F. Aleskerov, Multicriterial interval choice models, Inf. Sci. 80 (1/2) (1994) 25–41.

[7] F. Aleskerov, V. Vol'skiy, Choice of the best variants on binary relations and the extremizational choice, Preprints 10th World Congress on Automatic Control, 1987.

[8] W.E. Amstrong, The determinateness of the utility function, Econ. J. 49 (1939) 453–467.

[9] W.E. Amstrong, Uncertainty and the utility function, Econ. J. 58 (1948) 1–10.

[10] W.E. Amstrong, Utility and the theory of welfare, Oxford Econ. Paper 3 (1951) 259–271.

[11] J.P. Doignon, B. Monjardet, M. Roubens, Ph. Vincke, Biorder families, valued relations, and preference modelling, J. Math. Psychol. 30 (1986) 359–390.

[12] G.T. Fechner, Elemente der Psychophysik, Breitkopf und Hartel, Leipzig, 1860.

[13] P.C. Fishburn, Interval Orders and Interval Graphs, a Study of Partially Ordered Sets, Wiley, New York, 1985.

[14] J.C. Fodor, M. Roubens, Parametrized preference structure and some geometrical interpretation, J. Multi-Crit. Decision Anal. 6 (1997) 253–258.

[15] R.D. Luce, Semi-orders and a theory of utility discrimination, Econometrica 24 (1956) 78–191.

[16] B. Mirkin, Group Choice, Winston, New York, 1979.

[17] M. Pirlot, Ph. Vincke, Semiorders, Properties, Representations, Applications, Kluwer Academic Publishers, Dordrecht, 1997.

[18] L. Sholomov, D. Yudin, Synthesis of multi-step choice schemes, Autom. Remote Control 10 (1986) 115–126.

[19] N. Wiener, A contribution to the theory of relative position, Proc. Camb. Philos. Soc. 17 (1914).